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THEORETICAL STUDY OF TURBULENT SALINE DIFFUSION IN A CONDUIT IN CONTINUOUS SERVICE

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16637

ABSTRACT

It is possible to solve the equation of diffusion in a pipe by means of approximation hypotheses. The solutions obtained are either polynomials or Bessel functions, depending on the form of the diffusion coefficient that is adopted. The particular values are close in both cases.

author

During an experimental study of distributions of concentrations and their evolution in continuous operation, we came to investigate a diffusion diagram according to the classical model of the Reynolds analogy. The equation is written in reduced variables, the lengths being proportioned to the radius of the conduit a, and the velocity to the frictional velocity U_x.

$$\frac{1}{z}\frac{\partial}{\partial z}\left(k_1 z\frac{\partial c}{\partial z}\right)+\frac{1}{z}\frac{\partial}{\partial \theta}\left(\frac{k_2}{z}\frac{\partial c}{\partial \theta}\right)+\frac{\partial}{\partial \xi}\left(k_3\frac{\partial c}{\partial \xi}\right)=\frac{u(z)}{U_x}\frac{\partial c}{\partial \xi}. \tag{1}$$

in which

x = the abscissa measured along the axis of the conduit: $\xi = x/a$
r = polar radius: $z = r/a$
θ = polar angle
U = average velocity
$U/n = U_x$, frictional velocity

$u(z)$ = local velocity
k = constant of the logarithmic law of velocities
k_1, k_2, k_3 = reduced diffusion coefficients in the directions z, θ, ξ

c = local concentration relative to the average dilution concentration.

The Reynolds analogy makes it possible to define k_1 on the basis of a given law of deficient velocity. The simplest hypothesis is to suppose

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that $k_2 = k_1$; the longitudinal diffusion k_3 is then neglected before the convection term. Finally, a uniform distribution of velocities is assumed, an admissible assumption for distributions of concentration with slow damping.

Consider the case of inadequate speed in logarithmic form; $k_1 = kz(1-z)$, equation (1) becomes /2

$$\frac{1}{z}\frac{\partial}{\partial z}\left[z^2(1-z)\frac{\partial c}{\partial z}\right] + \frac{1-z}{z}\frac{\partial^2 c}{\partial \theta^2} = \frac{n}{k}\frac{\partial c}{\partial \xi}. \qquad (2)$$

The use of Fourier's method will result in solutions of the form

$$c = (\cos m\theta + \sin m\theta)\, e^{-\beta\xi} f(z) \qquad (3)$$

m is a positive integer and f is resolved in

$$z(1-z)f'' + (2-3z)f' + \left(s + m^2 - \frac{m^2}{z}\right)f = 0 \quad \text{with} \quad s = \frac{\beta n}{k}. \qquad (4)$$

Function f must be finite in the interval $0 \leqq z \leqq 1$. Fuchs' conditions obtain at points z = 0 and z = 1; r being the positive root of the determinant equation $r^2 + r - m^2 = 0$, the only finite solution z = 0 being written

$$f(z) = \sum_{p=0}^{p=\infty} a_p z^{p+r}. \qquad (5)$$

The coefficients a_p establish the equation of recurrence

$$a_{p+1} = a_p \frac{p(p+2) + r(2p+1) - s}{(p+1)(p+2) + 2r(p+1)}. \qquad (6)$$

For z = 1, the series (5) is divergent, except that if $s = s_{mp} = p(p+2) + (2p+1)r$, a_{p+1} is then zero, as are all the following coefficients. To each of these particular values there corresponds a polynomial f_{mp} that satisfies the conditions of the problem. In the particular case in which m = 0, the distributions are of revolution, equation (4) is that of Gauss, and the polynomials f_{0p} are Jacobi's polynomials.

The f_{mp} polynomials are orthogonal. Indeed, $\int_0^1 f_{mp} \cdot f_{mq} \cdot z \cdot dz = 0.$

Let us now consider the case in which the diffusion coefficient is constant across the section. Taking for the value of k_1 the average value of $k \cdot z(1-z)$, or $k/6$, the diffusion equation then becomes

$$\frac{\partial^2 c}{\partial z^2} + \frac{1}{z}\frac{\partial c}{\partial z} + \frac{1}{z^2}\frac{\partial^2 c}{\partial \theta^2} = \frac{6n}{k}\frac{\partial c}{\partial \xi}, \tag{7}$$

and the solutions are

$$C = (b_{mq} \cos m\theta + c_{mq} \sin m\theta)\, e^{-\beta_{mq}\xi} . J_m\left(z\sqrt{6 s_{mq}}\right)$$

the roots of the expression $J'_m\left(\sqrt{6 s_{mq}}\right)$, which expresses the condition at the limits, furnishing the proper values s_{mq}.

The proper values s_{mp} are given by equation

$$s_{mp} = p(p+2) + \frac{2p+1}{2}\left(\sqrt{1+4m^2} - 1\right). \tag{8}$$

It can be seen in the following table that they are close to the proper values s_{mq}: /3

	$p,q.$ $m \ldots$	0.	1.	2.	3.	4.
s_{mp}	0..........	0	0,618	1,561	2,541	3,531
	1..........	3	4,854	7,863	10,62	13,59
	2..........	8	11,09	15,81	20,71	25,66
	3..........	15	19,33	25,93	32,79	39,72
s_{mq}	0..........	0	0,565	1,860	2,94	4,713
	1..........	2,44	4,737	7,495	10,71	14,36
	2..........	8,21	12,14	16,56	21,46	26,80
	3..........	17,24	22,838	28,91	35,45	42,47

Note: commas in this table represent decimal points.

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